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Note

Conjectures on the enumeration of tableaux of bounded height ☆

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Abstract

We express general conjectures for the explicit form of P-recurrences for the number of Young standard tablaux of height bounded by h. These recurrences are shown to be compatible with known results and Regev's asymptotic evaluations.

1. Introduction

Let us first fix some notation. A partition λ of a positive integer n is a sequence of integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$ such that $\sum_i \lambda_i = n$. We write $\lambda \vdash n$ to express this fact, and denote $l(\lambda) = k$ the number of parts (the λ_i's) of λ. We say that k is the *height* $h(\lambda)$ of λ. The height of the empty partition (of 0) is set to be 0. The (Ferrer's) diagram of a partition is the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \leqslant j \leqslant \lambda_i$.

A standard Young tableau T is an injective labeling of a Ferrer's diagram by the elements of $\{1,2,\ldots,n\}$, such that $T(i,j) < T(i+1,j)$, for $1 \leqslant i < k$, and $T(i,j) < T(i,j+1)$, for $1 \leqslant j < \lambda_i$. We say further that λ is the *shape* of the tableau T. For a given λ, the number f_λ of tableaux of shape λ is given by the *hook length* formula

$$f_\lambda = \frac{n!}{\prod_c h_c},$$

where $c = (i,j)$ runs over the set of points in the diagram of λ, and

$$h_c = |\{(i,m) \in \lambda \mid m \geqslant j\} \cup \{(m,j) \in \lambda \mid m \geqslant i\}|.$$

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Other classical results in this context are

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!,$$

and the total number $t(n)$ of tableaux of *size* n is

$$t(n) = \sum_{\lambda \vdash n} f_\lambda = \text{coeff of } \frac{x^n}{n!} \text{ in } e^{x+x^2/2}.$$

We are interested in tableaux of height bounded by some integer h, this is to say that we want to compute the numbers

$$t_h(n) = \sum_{h(\lambda) \leqslant h} f_\lambda, \tag{1}$$

as well as

$$t_h^{(2)}(n) = \sum_{h(\lambda) \leqslant h} f_\lambda^2. \tag{2}$$

For the cases $h = 2, 3, 4, 5$ (see Regev [2] and Gouyou-Beauchamps [1]), nice expressions have been given for the $t_h(n)$'s or their generating function. We also should mention at this point that Zeilberger, in [3], has shown that the $t_h(n)$'s are P-recursive, this is to say that they satisfy a recurrence of the form

$$\sum_{k=0}^{m} p_k(n)\, t_h(n-k) = 0, \tag{3}$$

for some polynomials $p_k(n)$ and some integer m. Still, his proof gives no information on the bounds for m or the degrees of the $p_k(n)$'s. We propose, in this note, explicit values for the degrees of the polynomials appearing in (3) as well as for the value of m.

2. Conjectures for $t_h(n)$

Using the first values of the numbers $t_h(n)$ for small h's, and an undetermined coefficient method, we searched for simple P-recurrences for these numbers. The surprising outcome of these experiments was that these recurrences were of relatively low degree. A careful study of the first of these recurrences led us to the following conjectures. We then predicted the form of the recurrences for larger h's using these conjectures. Further computations showed that conjectured recurrences agreed with those given by an undetermined coefficient method. Our first conjectures state that:

(1a) The numbers $t_h(n)$ satisfy a recurrence of the form

$$\sum_{k=0}^{\lfloor h/2 \rfloor + 1} p_k(n)\, t_h(n-k) = 0, \quad (n > h), \tag{4}$$

with each polynomials $p_k(n)$ of degree $\leqslant \lfloor h/2 \rfloor$, and initial conditions $t_h(n)=t(n)$ for $n \leqslant h$.

(1b) The coefficient of $t_h(n)$ in (4) is

$$p_0(n)=\prod_{k=1}^{\lfloor h/2 \rfloor} (n+k(h-k)).$$

(1c) The coefficients of the $t_h(n-k)$'s in (4), $1 \leqslant k \leqslant \lfloor h/2 \rfloor$, are of the form

$$p_k(n)=q_k(n)\prod_{i=1}^{k-1}(n-i), \tag{5}$$

with the $q_k(n)$'s polynomials of respective degrees $\leqslant (\lfloor h/2 \rfloor - k + 1)$.

(1d) The polynomials $q_k(n)$ of (5) are such that recurrence (4) is true for all $n>0$, with the simple initial condition $t_h(0)=1$.

(1e) For odd h, the coefficient of $t_h(n-1)$ in (4) is

$$-p_1(n)=np_0(n)-(n-1)p_0(n-1)$$

and the leading coefficient of $q_k(n)$ is the coefficient of z^k in the polynomial

$$\prod_{j=0}^{m}(1-(-1)^{(m-j)}(2j+1)z),$$

where $h=2m+1$. Also the degree of $q_k(n)$ is exactly $m-k+1$.

Using these conjectures and an indeterminate coefficients method, we obtain the following recurrences for $h=1,3,5,7$ (the case $h=1$ is trivial).

$$t_1(n)=t_1(n-1),$$

$$(n+2)t_3(n)=(2n+1)t_3(n-1)+3(n-1)t_3(n-2),$$

$$\begin{aligned}(n+4)(n+6)t_5(n)=&(3n^2+17n+15)t_4(n-1)+(n-1)(13n+9)t_5(n-2)\\ &-15(n-1)(n-2)t_5(n-3),\end{aligned}$$

$$\begin{aligned}(n+6)(n+10)(n+12)t_7(n)=&(4n^3+78n^2+424n+495)t_7(n-1)\\ &+(n-1)(34n^2+280n+305)t_7(n-2)\\ &-(n-1)(n-2)(76n+290)t_7(n-3)\\ &-105(n-1)(n-2)(n-3)t_7(n-4).\end{aligned}$$

And for $h=2,4,6$,

$$(n+1)t_2(n)=2t_2(n-1)+4(n-1)t_2(n-2),$$

$$(n+3)(n+4)t_4(n)=4(3+2n)t_4(n-1)+16n(n-1)t_4(n-2),$$

$$\begin{aligned}(n+5)(n+8)(n+9)t_6(n)&=4(84+46n+5n^2)t_6(n-1)\\&\quad+4(n-1)(10n^2+58n+33)t_6(n-2)\\&\quad-144(n-1)(n-2)t_6(n-3)\\&\quad-144(n-1)(n-2)(n-3)t_6(n-4).\end{aligned}$$

Recurrences for bigger h's are easy to obtain in the same manner. But, the computation time gets to be quite large for $h\simeq 20$. We have checked that these recurrences are consistent with explicit computation (using (1)) of the $t_h(n)$ as far as reasonable computation time allowed ($n\simeq 40$). Moreover, for very large values of n ($n\simeq 2000$), the values of $t_h(n)$, obtained through (4), are strikingly consistent with the asymptotic expressions given by Regev in [2]. It is easy to show that the solution of a recurrence satisfying conjectures (1a)–(1d) is asymptotic to h^n, and a little extra work shows that the asymptotic behavior of the solution of the recurrence obtained with these conjectures is (see [2])

$$cte\,\frac{n^n}{n^{h(h-1)/4}},$$

for some constant cte. In fact, a simple translation of (4) in term of a differential equation for the generating function: $y(x)=\sum_{n\geqslant 0} t_h(n)x^n$, gives the following, for $h=7$,

$$\begin{aligned}&(1-7x)(1+5x)(1-3x)(1+x)x^3\,\frac{\mathrm{d}^3}{\mathrm{d}x^3}\,y(x)\\&\quad+(31-102x-552x^2+974x^3+945x^4)x^2\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,y(x)\\&\quad+(281-686x-1901x^2+2528x^3+1890x^4)x\,\frac{\mathrm{d}}{\mathrm{d}x}\,y(x)\\&\quad+(720-1001x-1001x^2+1036x^3+630x^4)y(x)=720.\end{aligned}$$

From this differential equation we easily find the (regular) singularity of smallest modulus of $y(x)$ since it is a root of the dominating polynomial. Using $y(x)\sim(1/7-x)^r$, we solve for r and find $r=19/2$ thus

$$t_h(n)\sim cte\,\frac{7^n}{n^{21/2}},\qquad \text{as } n\to\infty,$$

since the asymptotic behavior of the coefficients of $(1-hx)^r$ is some constant times h^n/n^{r+1}.

For odd $h=2m+1$, we have also obtained the following candidate for the generating function $\sum_m c_m x^m$ of coefficients c_m of n^{m-2} in the polynomial $g_2(n)$ of (5) (of degree $m-1$)

(1f) One has the generating function

$$\sum_m c_m x^m = -\frac{9x^2+217x^3+91x^4+3x^5}{(1-x)^7}.$$

Recall that Conjecture (1e) implicitly gives the coefficient of n^{m-1} in these polynomials. It appears that similar generating functions can be found for all coefficients of the $q_k(n)$'s. The $g_2(n)$ for $h=3,5,7,9$ are

$$-3$$

$$-13n-9$$

$$-34n^2-280n-305$$

$$-70n^3-1862n^2-13433n-18991.$$

3. Conjectures for $t_h^{(2)}(n)$

(2a) The numbers $t_h^{(2)}(n)$ satisfy a recurrence of the form

$$\sum_{k=0}^{\lfloor h/2\rfloor+1} p_k(n)t_h^{(2)}(n-k)=0, \quad (n>h), \tag{6}$$

with each polynomials $p_k(n)$ of degree $\leqslant n$, and initial conditions $t_h^{(2)}(n)=n!$ for $n\leqslant h$.

(2b) The coefficient of $t_h^{(2)}(n)$ in (6) is

$$p_0(n)=\prod_{k=1}^{\lfloor h/2\rfloor}(n+k(h>k))^2.$$

(2c) The coefficients of the $t_h^{(2)}(n-k)$'s in (6), $1\leqslant k\leqslant\lfloor h/2\rfloor+1$, are of the form

$$p_k(n)=q_k(n)\prod_{i=1}^{k-1}(n-i)^2, \tag{7}$$

with the $q_k(n)$'s polynomials of respective degrees $\leqslant(h-2k)$,

(2d) The polynomials $q_k(n)$ of (7) are such that recurrence (6) is true for all $n>0$, with the simple initial condition $t_h^{(2)}(0)=1$.

(2e) The leading coefficient of $q_k(n)$ is the coefficient of z^k in the polynomial

$$\prod_{j=0}^{m} (1-(2j+1)^2 z,$$

where $h=2m+1$.

All remarks that we have made about the $t_h(n)$'s also apply to the $t_h^{(2)}(n)$'s with the necessary modifications. For example, for $h=5$, the recurrence is

$$\begin{aligned}(n+4)^2(n+6)^2 t_h^{(2)}(n) &= (35n^4+322n^3+843n^2+400n-375)\,t_h^{(2)}(n-1)\\ &\quad -(n-1)^2(259n^2+622n+45)\,t_h^{(2)}(n-2)\\ &\quad +225(n-1)^2(n-2)^2\,t_h^{(2)}(n-3).\end{aligned}$$

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References

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